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Blandine Bérard-Bergery, Pierre P. Vallois

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Bérard Bergery Blandine, Vallois Pierre

IECN, Nancy-Université, CNRS, INRIA, Boulevard des Aiguillettes B.P. 239
F-54506 Vandœuvre lès Nancy

Abstract

We consider the convergence of the approximation schemes related to Itô's integral and quadratic variation, which have been developed in [8]. First, we prove that the convergence in the a.s. sense exists when the integrand is Hölder continuous and the integrator is a continuous semimartingale. Second, we investigate the second order convergence in the Brownian motion case.

Key words: stochastic integration by regularization, quadratic variation, first and second order convergence, stochastic Fubini's theorem
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1 Introduction

We consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, which satisfies the usual hypotheses. The notation (ucp) will stand for the convergence in probability, uniformly on the compact set in time.

1. Let X be a real continuous $(\mathcal{F}_t)$-semimartingale. In the usual stochastic calculus, the quadratic variation and the stochastic integral with respect to X play a central role. In [5], [6] and [7], Russo and Vallois extended these notions to continuous processes. Let us briefly recall their main definitions.

Definition 1.1 *Let X be a real-valued continuous process, $(\mathcal{F}_t)$-adapted, and H be a locally integrable process. The forward integral $\int_0^t H d^- X$ is defined as*

$$\int_0^t H d^- X = \lim_{\epsilon \to 0} (ucp) \, \frac{1}{\epsilon} \int_0^t H_u \left(X_{u+\epsilon} - X_u \right) du,$$

if the limit exists. The quadratic variation is defined by

$$[X]_t = \lim_{\epsilon \to 0} (ucp) \frac{1}{\epsilon} \int_0^t \left(X_{u+\epsilon} - X_u \right)^2 du$$

if the limit exists.

In the article, X will stand for a real-valued continuous $(\mathcal{F}_t)$-semimartingale and $(H_t)_{t\geqslant 0}$ for a $(\mathcal{F}_t)$-adapted process. If H is continuous, then, according to Proposition 1.1 of [5], the limits in (1.1) exist and coincide with the usual objects. In order to work with adapted processes only, we change $u+\epsilon$ into $(u+\epsilon)\wedge t$ in the integrals. This change does not affect the limit by (3.3) of [8]. Consequently,

$$\int_0^t H_u dX_u = \lim_{\epsilon\to 0}(ucp)\,\frac{1}{\epsilon}\int_0^t H_u\left(X_{(u+\epsilon)\wedge t} - X_u\right)du, \tag{1.1}$$

and

$$< X >_t = \lim_{\epsilon\to 0}(ucp)\frac{1}{\epsilon}\int_0^t \left(X_{(u+\epsilon)\wedge t} - X_u\right)^2 du \tag{1.2}$$

where $\int_0^t H_u dX_u$ is the usual stochastic integral and $< X >$ is the usual quadratic variation.

2. First, we determine sufficient conditions under which the convergences in (1.1) and (1.2) hold in the almost sure sense. Let us mention that some results in this direction have been obtained in [2].

We say that a process Y is *locally Hölder continuous* if, for all $T>0$, there exist $\alpha'\in]0,1]$ and $C_T\in L^2(\Omega)$ such that

$$|Y_s - Y_u| \leqslant C_T|u-s|^{\alpha'} \quad \forall u,s\in[0,T],\ \text{a.s.}$$

Our first result related to stochastic integral is the following.

Theorem 1.2 *If $(H_t)_{t\geqslant 0}$ is adapted and locally Hölder continuous, then*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t H_u(X_{(u+\epsilon)\wedge t} - X_u)du = \int_0^t H_u dX_u, \tag{1.3}$$

in the sense of almost sure convergence, uniformly on the compact sets in time.

Proposition 1.3 *If X is locally Hölder continuous, then*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du =< X >_t,$$

in the sense of almost sure convergence, uniformly on the compact sets in time. Moreover, if H is a continuous process,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t H_u(X_{(u+\epsilon)\wedge t} - X_u)^2 du = \int_0^t H_u d< X >_u, \tag{1.4}$$

in the sense of almost sure convergence.

3. Let us now consider the case where X is the standard Brownian motion B. Since B is a locally Hölder continuous martingale, the conditions in Theorem 1.2 and Proposition 1.3 are fulfilled. Then, it seems natural to determine the rate of convergence in (1.3) and (1.4), i.e. a second order convergence. Note that in [2], some related results have been proven.

Let us consider

$$\Delta_\epsilon(H,t) = \frac{1}{\sqrt{\epsilon}}\left[\frac{1}{\epsilon}\int_0^t H_u(B_{(u+\epsilon)\wedge t} - B_u)du - \int_0^t H_u dB_u\right], \tag{1.5}$$

$$\Delta_\epsilon^{(2)}(H,t) = \frac{1}{\sqrt{\epsilon}}\left[\frac{1}{\epsilon}\int_0^t H_u(B_{(u+\epsilon)\wedge t} - B_u)^2 du - \int_0^t H_u du\right], \tag{1.6}$$

where H is a progressively measurable process such that $\int_0^t H_s^2 ds < \infty$, for every $t > 0$.

We begin with $H_t = B_t$. In this case, we have:

$$\Delta_\epsilon(H,t) = W_\epsilon(t) + R_\epsilon(t),$$

where

$$W_\epsilon(t) = \int_0^t G_\epsilon(u)dB_u, \qquad G_\epsilon(u) = \frac{1}{\epsilon\sqrt{\epsilon}}\int_{(u-\epsilon)^+}^u (B_u - B_s)ds, \tag{1.7}$$

and

$$R_\epsilon(t) = \frac{1}{\sqrt{\epsilon}}\int_0^t \left(\frac{1}{\epsilon}\int_0^u B_s ds - \frac{u-\epsilon}{\epsilon}B_u\right)dB_u.$$

It is easy to verify that $R_\epsilon(t) \to 0$, as $\epsilon \to 0$, in $L^2(\Omega)$, and therefore does not contribute to the limit.

Theorem 1.4 *$(W_\epsilon(t), B_t)_{t\geqslant 0}$ converges in distribution to $(\sigma W_t, B_t)_{t\geqslant 0}$, as $\epsilon \to 0$, where W is a standard Brownian motion, independent from B, and $\sigma^2 = \frac{1}{3}$.*

We now investigate the convergence of $(\Delta_\epsilon(H,t))_{t\geqslant 0}$. We restrict ourselves to processes H of the type $H_t = H_0 + M_t + V_t$ where

1. H_0 is $\mathcal{F}_0$-measurable,
2. M_t is a Brownian martingale, i.e. $M_t = \int_0^t \Lambda_s dB_s$, where (Λ_t) is progressively measurable and satisfies $\int_0^t \Lambda_s^2 ds < \infty$,
3. V is a continuous process, which is Hölder continuous with order $\alpha > 1/2$, vanishing at time 0.

Note that if $V_t = \int_0^t \Phi_s ds$, where $(\Phi_t)_{t\geqslant 0}$ is adapted and locally bounded, then (3) holds with $\alpha = 1$ and H_t is a semimartingale.

Using a functional theorem of convergence (Proposition 3.2 and Theorem 5.1 in [3]) and Theorem 1.4, we obtain the following result.

Theorem 1.5 *1. For any $0 < t_1 < \cdots < t_n$, the random vector $(\Delta_\epsilon(H_0,t_1), \dots, \Delta_\epsilon(H_0,t_n))$ converges in law to $(\sigma H_0 N_0, \dots, \sigma H_0 N_0)$, where N_0 is a standard Gaussian r.v, independent from $\mathcal{F}_0$.*

2. *If V is a process which is locally Hölder continuous of order $\alpha > \frac{1}{2}$, then $\Delta_\epsilon(V,t)$ converges to 0 in $L^2(\Omega)$ as $\epsilon \to 0$, uniformly on the compact set in time.*

3. *If $M_t = \int_0^t \Lambda_s dB_s$, where $\Lambda_s = f(B_u, u \leqslant s)$ is a continuous function of the trajectory $(B_u, u \leqslant s)$ such that $t \to \Lambda_t$ is a continuous map from $\mathbb{R}^+$ to $L^2(\Omega)$, then the process $(\Delta_\epsilon(M,t))_{t\geqslant 0}$ converges in distribution to $(\sigma \int_0^t \Lambda_u dW_u)_{t\geqslant 0}$ as $\epsilon \to 0$.*

4. *If $H_0 = 0$, M and V are as in point $(2)-(3)$, then $(\Delta_\epsilon(H,t))_{t\geqslant 0}$ converges in law to $(\sigma \int_0^t \Lambda_u dW_u)_{t\geqslant 0}$ as $\epsilon \to 0$.*

The conditions of Theorem 1.5 related to the process H are likely too strong. In particular, the continuity of $t \to H_t$ and the fact that H is a semimartingale are not needed. Indeed, there exist adapted stepwise processes H so that $\Delta_\epsilon(H,t)$ converges in distribution as $\epsilon \to 0$, for any $t > 0$. More precisely, we have the following result.

Theorem 1.6 *Let $(a_i)_{i\in\mathbb{N}}$ be an increasing sequence of real numbers which satisfies $a_0 = 0$ and $a_n \to \infty$. Let $h, (h_i)_{i\in\mathbb{N}}$ be r.v.'s such that h_i is $\mathcal{F}_{a_i}$-measurable, h is $\mathcal{F}_0$-measurable and $\sup_{i\in\mathbb{N}} \|h_i\|_\infty < \infty$. Let H be the adapted and stepwise process:*

$$H_t = h \mathbb{1}_{\{t=0\}} + \sum_{i\geqslant 0} h_i \mathbb{1}_{\{t\in]a_i, a_{i+1}]\}}.$$

Then,

1. *$\frac{1}{\epsilon}\int_0^t H_s(B_{(s+\epsilon)\wedge t} - B_s)ds$ converges almost surely to $\int_0^t H_s dB_s$, uniformly on the compact set in time, as $\epsilon \to 0$.*

2. *There exists a sequence of i.i.d. r.v's $(N_i)_{i\in\mathbb{N}}$ with Gaussian law $\mathcal{N}(0,1)$, independent from B such that the r.v. $\Delta_\epsilon(H,t)$ converges in law to*

$$\frac{h_0}{\sqrt{3}} N_0 + \sum_{i\geqslant 1} \frac{(h_i - h_{i-1})}{\sqrt{3}} N_i \mathbb{1}_{\{t\leqslant a_{i+1}\}},$$

as $\epsilon \to 0$, for fixed time t.

A proof of Theorem 1.6 can be found in Section 6.3 of [1].

4. Let us finally present our second order result of convergence related to quadratic variation.

Proposition 1.7 *Let $H_s = f(B_u, u \leqslant s)$ be a continuous function of the trajectory $(B_u, u \leqslant s)$ such that $s \to H_s$ is locally Hölder continuous. Then, $(\Delta_\epsilon^{(2)}(H,t))_{t\geqslant 0}$ converges in distribution to $(\sigma \int_0^t H_u dW_u)_{t\geqslant 0}$, as $\epsilon \to 0$.*

5. Let us briefly detail the organization of the paper. Section 2 contains the proofs of the almost convergence results, i.e. Theorem 1.2 and Proposition 1.3. Then, the proof of Theorem 1.4 (resp. Proposition 1.7 and Theorem 1.5) is (resp. are) given in Section 3 (resp. Section 4).

In the calculations, C will stand for a generic constant (random or not). If C is random, then $C \in L^2(\Omega)$. We will use several times a stochastic version of Fubini's Theorem, which can be found in Section IV.5 of [4].

2 Proof of Theorem 1.2 and Proposition 1.3

We begin with the proof of Theorem 1.2 in Points **1-4** below. Then, we deduce Proposition 1.3 from Theorem 1.2 in Point **5**.

1. Let $T > 0$. We suppose that $(H_t)_{t\geqslant 0}$ is locally Hölder continuous of order α' and we study the almost sure convergence of

$$I_\epsilon(t) := \frac{1}{\epsilon}\int_0^t H_u(X_{(u+\epsilon)\wedge t} - X_u)du \text{ to } I(t) := \int_0^t H_u dX_u,$$

as $\epsilon \to 0$, uniformly on $t \in [0,T]$.

By stopping, we can suppose that $(X_t)_{0\leqslant t\leqslant T}$ and $< X >_T$ are bounded by a constant, and there exists a constant $C > 0$ such that

$$\forall u, s \in [0,T], \quad |H_s - H_u| \leqslant C|u-s|^\alpha, \quad \text{for some } \alpha \in]0, \alpha'[. \tag{2.1}$$

Let $X = X_0 + M + V$ be the canonical decomposition of X, where M is a continuous local martingale and V is an adapted process with finite variation. It is clear that $I_\epsilon(t) - I(t)$ can be decomposed as

$$\begin{aligned} I_\epsilon(t) - I(t) &= \left(\frac{1}{\epsilon}\int_0^t H_u(M_{(u+\epsilon)\wedge t} - M_u)du - \int_0^t H_u dX_u\right) \\ &\quad + \left(\frac{1}{\epsilon}\int_0^t H_u(M_{(u+\epsilon)\wedge t} - M_u)du - \int_0^t H_u dX_u\right). \end{aligned}$$

Theorem 1.2 will be proved as soon as $I_\epsilon(t) - I(t)$ converges to 0, in the case where X is either a continuous local martingale or a continuous finite variation process.

We deal with the finite variation case in Point **2**. As for the martingale case, the study is divided in two steps:

1. First, we prove that there is a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ such that $I_{\epsilon_n}(t)$ converges almost surely to $I(t)$ and $\epsilon_n \to 0$ (see Point **3** below).

2. Second, we show that $I_\epsilon(t)$ converges almost surely to 0, uniformly for $t \in [0,T]$ (see Point **4** below).

2. Suppose that X has a finite variation, writing $X_{(u+\epsilon)\wedge t} - X_u = \int_{(u+\epsilon)\wedge t}^{u} dX_s$ and using Fubini's theorem yield to:

$$\begin{aligned} I_\epsilon(t) - I(t) &= \int_0^t \left(\frac{1}{\epsilon} \int_{(s-\epsilon)^+}^{s} H_u du - H_s \right) dX_s, \\ &= \int_0^t \left(\frac{1}{\epsilon} \int_{(s-\epsilon)^+}^{s} (H_u - H_s) du \right) dX_s - \int_0^\epsilon \frac{\epsilon - s}{\epsilon} H_s dX_s. \end{aligned}$$

Using the Hölder property (2.1) (in the first integral) and the fact that H is bounded by a constant (in the second integral), we have for all $t \in [0,T]$:

$$\begin{aligned} |I_\epsilon(t) - I(t)| &\leqslant \int_0^T \left(\frac{1}{\epsilon} \int_{(s-\epsilon)^+}^{s} C|u-s|^\alpha du \right) d|X|_s + \int_0^\epsilon \frac{\epsilon - s}{\epsilon} C\, d|X|_s, \\ &\leqslant C\epsilon^\alpha |X|_T + C(|X|_\epsilon - |X|_0). \end{aligned}$$

Consequently, $I_\epsilon(t) - I(t)$ converges almost surely to 0, as $\epsilon \to 0$, uniformly on any compact set in time.

3. In the two next points, X is a continuous martingale. We proceed as in step **2** above: observing that $X_{(u+\epsilon)\wedge t} - X_u = \int_{(u+\epsilon)\wedge t}^{u} dX_s$ and using Fubini's stochastic theorem come to

$$I_\epsilon(t) - I(t) = \int_0^t \left(\frac{1}{\epsilon} \int_{(s-\epsilon)^+}^{s} H_u du - H_s \right) dX_s.$$

Thus, $(I_\epsilon(t) - I(t))_{t\in[0,T]}$ is a continuous local martingale. Moreover, $E(< I_\epsilon - I >_t)$ is bounded since H and $< X >$ are bounded on $[0,T]$.

Let us introduce $p = \frac{2(1-\alpha)}{\alpha^2} + 1$. This explicit expression of p in terms of α will be used later at the end of Point **4**. Burkholder-Davis-Gundy inequalities give:

$$E\left(\sup_{t\in[0,T]} |I_\epsilon(t) - I(t)|^p \right) \leqslant c_p E\left[\left(\int_0^T \left(\frac{1}{\epsilon} \int_{(s-\epsilon)^+}^{s} H_u du - H_s \right)^2 d < X >_s \right)^{\frac{p}{2}} \right].$$

The Hölder property (2.1) implies that:

$$\left| \frac{1}{\epsilon} \int_{(s-\epsilon)^+}^{s} H_u du - H_s \right| \leqslant \frac{1}{\epsilon} \int_{(s-\epsilon)^+}^{s} |H_u - H_s|\, du \leqslant C\epsilon^\alpha.$$

Consequently,

$$E\left(\sup_{t\in[0,T]}|I_\epsilon(t)-I(t)|^p\right)\leqslant C\epsilon^{\alpha p}E[<X>_T]^{\frac{p}{2}}\leqslant C\epsilon^{\alpha p}.$$

Then, for any $\delta>0$, Markov inequality leads to :

$$P\left(\sup_{t\in[0,T]}|I_\epsilon(t)-I(t)|>\delta\right)\leqslant\frac{C\epsilon^{\alpha p}}{\delta^p}. \tag{2.2}$$

Let us now define $(\epsilon_n)_{n\in\mathbb{N}^\star}$ by $\epsilon_n=n^{-\frac{2}{p\alpha}}$ for all $n>0$. Replacing ϵ by ϵ_n in (2.2) comes to:

$$P\left(\sup_{t\in[0,T]}|I_{\epsilon_n}(t)-I(t)|>\delta\right)\leqslant\frac{C}{\delta^p}n^{-2}.$$

Since $\sum_{n=1}^{\infty}n^{-2}<\infty$, the Borel-Cantelli lemma implies that:

$$\lim_{n\to\infty}\sup_{t\in[0,T]}|I_{\epsilon_n}(t)-I(t)|=0,\quad a.s. \tag{2.3}$$

4. For all $\epsilon\in]0,1[$, let $n=n(\epsilon)$ denote the integer such that $\epsilon\in]\epsilon_{n+1},\epsilon_n]$. Then, we decompose $I_\epsilon(t)-I(t)$ as follows:

$$I_\epsilon(t)-I(t)=(I_\epsilon(t)-I_{\epsilon_n}(t))+(I_{\epsilon_n}(t)-I(t)).$$

(2.3) gives the almost sure convergence of $I_{\epsilon_n}(t)$ to $I(t)$, uniformly on $[0,T]$. Therefore, the a.s convergence of $I_\epsilon(t)-I(t)$ to 0 , uniformly on $[0,T]$, will be obtained as soon as $I_\epsilon(t)-I_{\epsilon_n}(t)$ goes to 0, uniformly on $[0,T]$.

From the definition of $I_\epsilon(t)$, it is easy to deduce that we have:

$$\begin{aligned}I_\epsilon(t)-I_{\epsilon_n}(t) &= \frac{1}{\epsilon}\left(\int_0^t H_u(X_{(u+\epsilon)\wedge t}du-\int_0^t H_uX_{(u+\epsilon_n)\wedge t})du\right)\\ &\quad+\left(\frac{1}{\epsilon}-\frac{1}{\epsilon_n}\right)\left(\int_0^t H_u(X_{(u+\epsilon_n)\wedge t}-X_u)du\right).\end{aligned}$$

The changes of variable either $v=u+\epsilon$ or $v=u+\epsilon_n$ lead to

$$\begin{aligned}I_\epsilon(t)-I_{\epsilon_n}(t) &= \frac{1}{\epsilon}\int_\epsilon^t\left(H_{v-\epsilon}-H_{v-\epsilon_n}\right)X_vdv \\ &\quad+\frac{\epsilon_n-\epsilon}{\epsilon\epsilon_n}\left(\int_{\epsilon_n}^t\left(H_{v-\epsilon_n}-H_v\right)X_vdv\right)+R_\epsilon(t),\end{aligned} \tag{2.4}$$

where we gather under the notation $R_\epsilon(t)$ all the remaining terms. Let us observe that $R_\epsilon(t)$ is the sum of terms which are of the form $\frac{1}{\epsilon}\int_a^b\dots dv$ where

$|a-b| \leqslant \epsilon_n - \epsilon$ or $\left(\frac{1}{\epsilon} - \frac{1}{\epsilon_n}\right) \int_a^b \dots dv$ where $|a-b| \leqslant \epsilon_n$. Since H and X are bounded on $[0,T]$, we have

$$|R_\epsilon(t)| \leqslant C\frac{\epsilon_n - \epsilon}{\epsilon} \quad \forall t \in [0,T]. \tag{2.5}$$

By the Hölder property (2.1), we get

$$|H_{v-\epsilon} - H_{v-\epsilon_n}| \leqslant C(\epsilon_n - \epsilon)^\alpha, \quad |H_{v-\epsilon_n} - H_v| \leqslant C\epsilon_n^\alpha. \tag{2.6}$$

Since X and H are bounded, we can deduce from (2.4), (2.5) and (2.6) that:

$$|I_\epsilon(t) - I_{\epsilon_n}(t)| \leqslant C\frac{(\epsilon_n - \epsilon)^\alpha}{\epsilon} + C\frac{(\epsilon_n - \epsilon)\epsilon_n^\alpha}{\epsilon\epsilon_n} + C\frac{\epsilon - \epsilon_n}{\epsilon}, \, \forall t \in [0,T].$$

Using the definition of ϵ_n, we infer

$$\frac{\epsilon_n - \epsilon}{\epsilon} \leqslant Cn^{-1}, \, \frac{(\epsilon_n - \epsilon)^\alpha}{\epsilon} \leqslant Cn^{\frac{2(1-\alpha)}{p\alpha} - \alpha}, \, \frac{(\epsilon_n - \epsilon)\epsilon_n^\alpha}{\epsilon\epsilon_n} \leqslant n^{-\frac{2}{p} - 1 + \frac{2}{p\alpha}} \leqslant n^{\frac{2(1-\alpha)}{p\alpha} - \alpha}.$$

Note that $p = \frac{2(1-\alpha)}{\alpha^2} + 1$ implies that $\frac{2(1-\alpha)}{p\alpha} - \alpha < 0$. As a result, $I_\epsilon(t) - I_{\epsilon_n}(t)$ goes to 0 a.s, uniformly on $[0,T]$, as $\epsilon \to 0$. ■

5. In this item, it is supposed that X is a semimartingale, locally Hölder continuous. It is clear that $\frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du$ equals

$$\frac{1}{\epsilon}\left[\int_0^t X^2_{(u+\epsilon)\wedge t} du - \int_0^t X_{(u+\epsilon)\wedge t} X_u du - \int_0^t X_u (X_{(u+\epsilon)\wedge t} - X_u) du\right].$$

Thanks to the change of variable $v = u + \epsilon$ in the first integral, after easy calculations, we get

$$\frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du = X_t^2 - \frac{1}{\epsilon}\int_0^\epsilon X^2_{v\wedge t} dv - \frac{2}{\epsilon}\int_0^t X_u(X_{(u+\epsilon)\wedge t} - X_u) du.$$

Since X is continuous, $\frac{1}{\epsilon}\int_0^\epsilon X^2_{v\wedge t} dv$ tends to X_0^2 a.s, uniformly on $[0,T]$. According to Theorem 1.2, we have

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\int_0^t (X_{(u+\epsilon)\wedge t} - X_u)^2 du = X_t^2 - X_0^2 - 2\int_0^t X_u dX_u.$$

Itô 's formula implies that the right-hand side of the above identity equals to $< X >_t$.

Replacing $(u+\epsilon)\wedge t$ by $(u+\epsilon)^+$ does not change the limit. Then, the measure $\frac{1}{\epsilon}(X_{(u+\epsilon)^+} - X_u)^2 du$ converges a.s. to the measure $d < X >_u$. It leads to the convergence a.s. of $\frac{1}{\epsilon}\int_0^t H_u (X_{(u+\epsilon)\wedge t} - X_u)^2 du$ to $\int_0^t H_u d < X >_u$, for H continuous process. ■

3 Proof of Theorem 1.4

Recall that $W_\epsilon(t)$ and $G_\epsilon(t)$ are defined by (1.7). We study the convergence in distribution of the two dimensional process $(W_\epsilon(t), B_t)$, as $\epsilon \to 0$.

First, we determine the limit in law of $W_\epsilon(t)$. In Point **1** we demonstrate preliminary results. Then, we prove the convergence of the moments of $W_\epsilon(t)$ in Point **2**. By the method of moments, the convergence in law of $W_\epsilon(t)$ for a fixed time is proven in Point **3**. We deduce the finite-dimensionnal convergence in Point **4**. Finally, Kolmogorov criterion concludes the proof in Point **5.** Then, we briefly sketch in Point **6** the proof of the joint convergence of $(W_\epsilon(t))_{t\geqslant 0}$ and $(B_t)_{t\geqslant 0}$. The approach is close to the one of $(W_\epsilon(t))_{t\geqslant 0}$.

1. We begin by calculating the moments of $W_\epsilon(t)$ and $G_\epsilon(u)$. We denote by $\stackrel{\mathcal{L}}{=}$ the equality in law.

Lemma 3.1 $E\left[|G_\epsilon(u)|^2\right] = \frac{(u\wedge\epsilon)^3}{\epsilon^3}\sigma^2$. *Moreover, for all* $k \in \mathbb{N}$, *there exists a constant* m_k *such that* $E\left[|G_\epsilon(u)|^k\right] \leqslant m_k$, $\forall u \geqslant 0, \epsilon > 0$.

Proof. First, we apply the change of variable $s = u - (u\wedge\epsilon)r$ in (1.7). Then, using the identity $(B_u - B_{u-v}; 0 \leqslant v \leqslant u) \stackrel{\mathcal{L}}{=} (B_v; 0 \leqslant v \leqslant u)$ and the scaling property of B, we get

$$G_\epsilon(u) \stackrel{\mathcal{L}}{=} \frac{(u\wedge\epsilon)\sqrt{u\wedge\epsilon}}{\epsilon\sqrt{\epsilon}} \int_0^1 B_r dr.$$

Since $\int_0^1 B_r dr \stackrel{\mathcal{L}}{=} \sigma N$, where $\sigma^2 = 1/3$ and N is a standard gaussian r.v, we obtain

$$E\left[|G_\epsilon(u)|^k\right] = \frac{(u\wedge\epsilon)^{\frac{3k}{2}}}{\epsilon^{\frac{3k}{2}}}\sigma^k E\left[|N|^k\right]. \tag{3.1}$$

Taking $k = 2$ gives $E\left[|G_\epsilon(u)|^2\right] = \frac{(t\wedge\epsilon)^3}{\epsilon^3}\sigma^2$. Using $u\wedge\epsilon \leqslant \epsilon$ and (3.1), we get $E[|G_\epsilon(u)|^k] \leqslant m_k$ with $m_k = \sigma^k E\left[|N|^k\right]$. ■

Lemma 3.2 *For all* $k \geqslant 2$, *there exists a constant* $C(k)$ *such that*

$$\forall t \geqslant 0, \quad E\left[|W_\epsilon(t)|^k\right] \leqslant C(k)\, t^{\frac{k}{2}}.$$

Moreover, for $k = 2$, *we have*

$$E\left[\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)^2\right] \leqslant \sigma^2\epsilon, \quad \forall u \geqslant 0.$$

Proof. The Burkhölder-Davis-Gundy inequality and (1.7) give

$$E\left[|W_\epsilon(t)|^k\right] \leqslant c(k) E\left[\left(\int_0^t (G_\epsilon(u))^2\, du\right)^{\frac{k}{2}}\right].$$

Then, Jensen inequality implies:

$$E\left[\left(\int_0^t (G_\epsilon(u))^2\,du\right)^{\frac{k}{2}}\right] \leqslant t^{\frac{k}{2}-1} E\left[\int_0^t |G_\epsilon(u)|^k\,du\right].$$

Finaly, applying Lemma 3.1 comes to

$$E\left[|W_\epsilon(t)|^k\right] \leqslant c(k) m_k t^{\frac{k}{2}}.$$

The case $k=2$ can be easily treated via (1.7) and Lemma 3.1:

$$\begin{aligned} E\left[\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)^2\right] &= \int_{(u-\epsilon)^+}^u E\left[(G_\epsilon(v))^2\right] dv, \\ &= \int_{(u-\epsilon)^+}^u \sigma^2 \frac{(v\wedge\epsilon)^3}{\epsilon^3} dv \leqslant \sigma^2\epsilon. \end{aligned}$$

■

2. Let us now study the convergence of the moments of $W_\epsilon(t)$.

Proposition 3.3

$$\lim_{\epsilon\to 0} E\left[(W_\epsilon(t))^{2n}\right] = E\left[(\sigma W_t)^{2n}\right], \quad \forall n\in\mathbb{N}, t\geqslant 0. \tag{3.2}$$

Proof. a) We prove Proposition 3.3 by induction on $n \geqslant 1$.

For $n=1$, from Lemma 3.1, we have:

$$E\left[(W_\epsilon(t))^2\right] = \int_0^t E\left[(G_\epsilon(u))^2\right] du = \int_0^t \sigma^2 \frac{(u\wedge\epsilon)^3}{\epsilon^3} du.$$

Then, $E\left[(W_\epsilon(t))^2\right]$ converges to $\sigma^2 t = E[(\sigma W_t)^2]$.

Let us suppose that (3.2) holds. First, we apply Itô's formula to $(W_\epsilon(t))^{2n+2}$. Second, taking the expectation reduces to 0 the martingale part. Finally, we get

$$E\left[(W_\epsilon(t))^{2n+2}\right] = \frac{(2n+2)(2n+1)}{2}\int_0^t E\left[(W_\epsilon(u))^{2n}(G_\epsilon(u))^2\right] du. \tag{3.3}$$

b) We admit for a while that

$$E\left[(W_\epsilon(u))^{2n}(G_\epsilon(u))^2\right] \longrightarrow \sigma^2 E\left[(\sigma W_u)^{2n}\right], \quad \forall u\geqslant 0. \tag{3.4}$$

Using Cauchy-Schwarz inequality and Lemmas 3.1, 3.2 give:

$$\begin{aligned} E\left[(W_\epsilon(u))^{2n}(G_\epsilon(u))^2\right] &\leqslant \sqrt{E\left[(W_\epsilon(u))^{4n}\right] E\left[(G_\epsilon(u))^4\right]} \\ &\leqslant \sqrt{C(4n)u^{2n}m_4} \leqslant \sqrt{C(4n)m_4}\,u^n. \end{aligned}$$

Consequently, we may apply Lebesgue's theorem to (3.3), we have

$$\begin{aligned}
\lim_{\epsilon\to 0} E\left[(W_\epsilon(t))^{2n+2}\right] &= \frac{(2n+2)(2n+1)}{2}\sigma^2 \int_0^t E\left[(\sigma W_u)^{2n}\right] du, \\
&= \frac{(2n+2)(2n+1)}{2}\sigma^{2n+2} \int_0^t u^n \frac{(2n)!}{n!\,2^n} du, \\
&= \frac{(2n+2)!}{(n+1)!\,2^{n+1}}(\sigma\sqrt{t})^{2n+2} = E\left[(\sigma W_t)^{2n+2}\right].
\end{aligned}$$

c) We have now to prove (3.4). If $u = 0$, $E\left[(W_\epsilon(0))^{2n}(G_\epsilon(0))^2\right] = 0 = \sigma^2 E\left[(\sigma W_0)^{2n}\right]$. If $u > 0$, it is clear that:

$$E\left[(W_\epsilon(u))^{2n}(G_\epsilon(u))^2\right] = E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n}(G_\epsilon(u))^2\right] + \xi_\epsilon(u), \qquad (3.5)$$

where

$$\xi_\epsilon(u) = E\left[\left\{(W_\epsilon(u))^{2n} - \left(W_\epsilon((u-\epsilon)^+)\right)^{2n}\right\}(G_\epsilon(u))^2\right].$$

Since $G_\epsilon(u)$ is independent from $\mathcal{F}_{(u-\epsilon)^+}$, we have

$$E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n}(G_\epsilon(u))^2\right] = E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n}\right] E\left[(G_\epsilon(u))^2\right].$$

Finally, plugging the identity above in (3.5) gives:

$$E\left[(W_\epsilon(u))^{2n}(G_\epsilon(u))^2\right] = E\left[(W_\epsilon(u))^{2n}\right] E\left[(G_\epsilon(u))^2\right] + \xi_\epsilon(u) + \tilde{\xi}_\epsilon(u),$$

where

$$\tilde{\xi}_\epsilon(u) = E\left[\left(W_\epsilon((u-\epsilon)^+)\right)^{2n} - (W_\epsilon(u))^{2n}\right] E\left[(G_\epsilon(u))^2\right].$$

Lemma 3.1 implies that $E\left[(G_\epsilon(u))^2\right]$ tends to σ^2 as $\epsilon \to 0$. The recurrence hypothesis implies that $E\left[(W_\epsilon(u))^{2n}\right]$ converges to $E\left[(\sigma W_u)^{2n}\right]$ as $\epsilon \to 0$. It remains to prove that $\xi_\epsilon(u)$ and $\tilde{\xi}_\epsilon(u)$ tend to 0 to conclude the proof.

The identity $a^{2n} - b^{2n} = (a-b)\sum_{k=0}^{2n-1} a^k b^{2n-1-k}$ implies that $\xi_\epsilon(u)$ is equal to the sum $\sum_{k=0}^{2n-1} S_k(\epsilon, u)$, where

$$S_k(\epsilon,u) = E\left[\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)(G_\epsilon(u))^2 (W_\epsilon(u))^k (W_\epsilon((u-\epsilon)^+))^{2n-1-k}\right].$$

Applying four times the Cauchy-Schwarz inequality comes to:

$$\begin{aligned}
|S_k(\epsilon,u)| &\leqslant \left[E\left(W_\epsilon(u) - W_\epsilon((u-\epsilon)^+)\right)^2\right]^{\frac{1}{2}} \left[E\left(G_\epsilon(u)\right)^8\right]^{\frac{1}{4}} \\
&\quad \times \left[E(W_\epsilon(u))^{8k}\right]^{\frac{1}{8}} \left[E(W_\epsilon((u-\epsilon)^+))^{16n-8-8k}\right]^{\frac{1}{8}}.
\end{aligned}$$

Lemmas 3.1 and 3.2 lead to

$$|S_k(\epsilon,u)| \leqslant C(k) T^{n-\frac{1}{2}}\sqrt{\epsilon}, \quad \forall u \in [0,T].$$

Consequently, $\xi_\epsilon(u)$ tends to 0 as $\epsilon \to 0$. Using the same method, it is easy to prove that $\tilde{\xi}_\epsilon(u)$ tends to 0 as $\epsilon \to 0$. ■

3. From Proposition 3.3, it easy to deduce the convergence in law of $W_\epsilon(t)$ (t being fixed).

Proposition 3.4 *For any fixed $t \geqslant 0$, $W_\epsilon(t)$ converges in law to σW_t, as $\epsilon \to 0$.*

Let us recall the method of moments.

Proposition 3.5 *Let $X, (X_n)_{n\in\mathbb{N}}$ be r.v's such that $E(|X|^k) < \infty$, $E(|X_n|^k) < \infty, \forall k, n \in \mathbb{N}$ and*

$$\overline{\lim}_{k\to\infty} \frac{[E(X^{2k})]^{\frac{1}{2k}}}{2k} < \infty. \tag{3.6}$$

If for all $k \in \mathbb{N}$, $\lim_{n\to\infty} E(X_n^k) = E(X^k)$, then X_n converges in law to X as $n \to \infty$.

Proof of Proposition 3.4. Let $t \geqslant 0$ be a fixed time. The odd moments of $W_\epsilon(t)$ are null. By Proposition 3.3, the even moments of $W_\epsilon(t)$ tends to σW_t. Since σW_t is a Gaussian r.v. with variance $\sigma\sqrt{t}$, it is easy to check that (3.6) holds. As a result, $W_\epsilon(t)$ converges in law to σW_t. ■

4. Next, we prove the finite-dimensionnal convergence.

Proposition 3.6 *Let $0 < t_1 < t_2 < \cdots < t_n$. Then, $(W_\epsilon(t_1), \ldots, W_\epsilon(t_n))$ converges in law to $(\sigma W_{t_1}, \ldots, \sigma W_{t_n})$, as $\epsilon \to 0$.*

Proof. We take $n = 2$ for simplicity. We consider $0 < t_1 < t_2$ and $\epsilon \in]0, t_1 \wedge (t_2 - t_1)[$. Since $t_1 > \epsilon$, note that $(u-\epsilon)^+ = u - \epsilon$ for $u \in [t_1, t_2]$. We begin with the decomposition:

$$W_\epsilon(t_2) = W_\epsilon(t_1) + \frac{1}{\epsilon\sqrt{\epsilon}} \int_{t_1+\epsilon}^{t_2} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u + R_\epsilon^1(t_1, t_2),$$

where $R_\epsilon^1(t_1,t_2) = \frac{1}{\epsilon\sqrt{\epsilon}} \int_{t_1}^{t_1+\epsilon} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u$. Let us note that $W_\epsilon(t_1)$ is independent from $\frac{1}{\epsilon\sqrt{\epsilon}} \int_{t_1+\epsilon}^{t_2} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u$.

Let us introduce $B'_t = B_{t+t_1} - B_{t_1}, t \geqslant 0$. B' is a standard Brownian motion. The changes of variables $u = t_1 + v$ and $r = s - t_1$ in $\int_{t_1+\epsilon}^{t_2} \left(\int_{u-\epsilon}^{u} (B_u - B_s) ds \right) dB_u$ leads to

$$W_\epsilon(t_2) = W_\epsilon(t_1) + \Theta_\epsilon(t_1, t_2) + R_\epsilon^2(t_1, t_2) + R_\epsilon^1(t_1, t_2), \tag{3.7}$$

where

$$\begin{aligned}
\Theta_\epsilon(t_1, t_2) &= \frac{1}{\epsilon\sqrt{\epsilon}} \int_0^{t_2 - t_1} \left(\int_{(v-\epsilon)^+}^{v} (B'_v - B'_r) dr \right) dB'_v, \\
R_\epsilon^2(t_1, t_2) &= \frac{1}{\epsilon\sqrt{\epsilon}} \int_0^{\epsilon} \left(\int_0^{v} (B'_v - B'_r) dr \right) dB'_v.
\end{aligned}$$

Straightforward calculation shows that $E\left[(R_\epsilon^1(t_1, t_2))^2 \right]$ and $E\left[(R_\epsilon^2(t_1, t_2))^2 \right]$ are bounded by $C\epsilon$. Thus, $R_\epsilon^1(t_1, t_2)$ and $R_\epsilon^1(t_1, t_2)$ converge to 0 in $L^2(\Omega)$.

Proposition 3.4 gives the convergence in law of $\Theta_\epsilon(t_1,t_2)$ to $\sigma(W_{t_2}-W_{t_1})$ and the convergence in law of $W_\epsilon(t_1)$ to σW_{t_1}, as $\epsilon \to 0$.

Since $W_\epsilon(t_1)$ and $\Theta_\epsilon(t_1,t_2)$ are independent, the decomposition (3.7) implies that $(W_\epsilon(t_1), W_\epsilon(t_2)-W_\epsilon(t_1))$ converges in law to $(\sigma W_{t_1}, \sigma(W_{t_2}-W_{t_1}))$, as $\epsilon \to 0$. Proposition 3.4 follows immediately. ■

5. We end the proof of the convergence in law of the process $(W_\epsilon(t))_{t\geqslant 0}$ by showing that the family of the laws of $(W_\epsilon(t))_{t\geqslant 0}$ is tight as $\epsilon \in]0,1]$.

Lemma 3.7 *There exists a constant K such that*

$$E\left[|W_\epsilon(t)-W_\epsilon(s)|^4\right] \leqslant K|t-s|^2, \qquad 0\leqslant s\leqslant t, \epsilon>0.$$

Proof. Applying Burkhölder-Davis-Gundy inequality, we obtain:

$$E\left[|W_\epsilon(t)-W_\epsilon(s)|^4\right] \leqslant cE\left[\left(\int_s^t (G_\epsilon(u))^2\,du\right)^2\right] \leqslant c(t-s)\int_s^t E\left[(G_\epsilon(u))^4\right] du.$$

Using Lemma 3.1, we get $E\left[|W_\epsilon(t)-W_\epsilon(s)|^4\right] \leqslant c\,m_4(t-s)^2$ and ends the proof (see Kolmogorov Criterion in Section XIII-1 of [4]). ■

6. To prove the joint convergence of $(W_\epsilon(t),B_t)_{t\geqslant 0}$ to $(\sigma W_t, B_t)_{t\geqslant 0}$, we mimick the approach developed in Points **1-5** above.

6.a. Convergence $(W_\epsilon(t),B_t)$ to $(\sigma W_t, B_t)$, t being fixed.
First, we prove that

$$\lim_{\epsilon\to 0} E(W_\epsilon^p(t)B_t^q) = E((\sigma W_t)^p B_t^q), \qquad p,q\in\mathbb{N}. \tag{3.8}$$

Let us note that the limit is null when either p or q is odd.

Using Itô's formula, we get

$$E\left[(W_\epsilon(t))^p B_t^q\right] = \frac{p(p-1)}{2}\alpha_1(t,\epsilon) + \frac{q(q-1)}{2}\alpha_2(t,\epsilon) + pq\alpha_3(t,\epsilon),$$

where

$$\begin{aligned}
\alpha_1(t,\epsilon) &= \int_0^t E\left[(W_\epsilon(u))^{p-2} B_u^q (G_\epsilon(u))^2\right] du,\\
\alpha_2(t,\epsilon) &= \int_0^t E\left[(W_\epsilon(u))^{p} B_u^{q-2}\right] du,\\
\alpha_3(t,\epsilon) &= \int_0^t E\left[(W_\epsilon(u))^{p-1} B_u^{q-1} G_\epsilon(u)\right] du.
\end{aligned}$$

To demonstrate (3.8), we proceed by induction on q, then by induction on p, q being fixed.
First, we apply (3.8) with $q-2$ instead of q, then we have directly:

$$\lim_{\epsilon\to 0}\alpha_2(t,\epsilon) = \int_0^t E\left[(\sigma W_u)^p\right] E\left[B_u^{q-2}\right] du.$$

As for $\alpha_1(t,\epsilon)$, we write

$$\begin{aligned} (W_\epsilon(u))^{p-2} &= (W_\epsilon(u))^{p-2} - \big(W_\epsilon((u-\epsilon)^+)\big)^{p-2} + \big(W_\epsilon((u-\epsilon)^+)\big)^{p-2} \\ B_u^q &= B_u^q - B^q_{(u-\epsilon)^+} + B^q_{(u-\epsilon)^+}. \end{aligned}$$

We proceed similarly with $\alpha_3(t,\epsilon)$. Reasoning as in Point **2** and using the two previous identities, we can prove:

$$\lim_{\epsilon\to 0}\alpha_1(t,\epsilon) = \sigma^2\int_0^t E\left[(\sigma W_u)^{p-2}\right]E\left[B_u^q\right]du \text{ and } \lim_{\epsilon\to 0}\alpha_3(t,\epsilon) = 0.$$

Consequently, when either p or q is odd, then $\lim_{\epsilon\to 0}\alpha_i(t,\epsilon) = 0$, $(i = 1,2)$ and therefore:

$$\lim_{\epsilon\to 0} E(W_\epsilon^p(t)B_t^q) = 0 = E((\sigma W_t)^p B_t^q).$$

It remains to determine the limit in the case where p and q are even. Let us denote $p = 2p'$ and $q = 2q'$. Then we have

$$\begin{aligned} \lim_{\epsilon\to 0}\alpha_1(t,\epsilon) &= \int_0^t \sigma^2\frac{(p-2)!}{2^{p'-1}(p'-1)!}u^{p'-1}\sigma^{p-2}\frac{q!}{2^{q'}(q')!}u^{q'}du \\ &= \frac{(p-2)!\,q!}{2^{p'+q'-1}(p'-1)!\,(q')!\,(p'+q')}\sigma^p t^{p'+q'}, \\ \lim_{\epsilon\to 0}\alpha_2(t,\epsilon) &= \int_0^t \frac{p!}{2^{p'}(p')!}\sigma^p u^{p'}\frac{(q-2)!}{2^{q'-1}(q'-1)!}u^{q'-1}du \\ &= \frac{p!\,(q-2)!}{2^{p'+q'-1}(p')!\,(q'-1)!\,(p'+q')}\sigma^p t^{p'+q'}. \end{aligned}$$

Then, it is easy to deduce

$$\lim_{\epsilon\to 0} E\left[(W_\epsilon(t))^p B_t^q\right] = \frac{p!}{2^{p'}(p')!}\sigma^p t^{p'}\frac{q!}{2^{q'}(q')!}t^{q'} = E\left[(\sigma W_t)^p\right]E\left[B_t^q\right].$$

Next, we use a two dimensional version of the method of moments:

Proposition 3.8 *Let $X, Y, (Y_n)_{n\in\mathbb{N}}(X_n)_{n\in\mathbb{N}}$ be r.v. whose moments are finite. Let us suppose that X and Y satisfy (3.6) and that $\forall p,q\in\mathbb{N}$, $\lim_{n\to\infty}E(X_n^pY_n^q) = E(X^pY^q)$. Then, (X_n,Y_n) converges in law to (X,Y) as $n\to\infty$.*

Since W_t and B_t are Gaussian r.v's, they both satisfy (3.6). Consequently, $(W_\epsilon(t), B_t)$ converges in law to $(\sigma W_t, B_t)$ as $\epsilon\to 0$.

6.b. Finite-dimensional convergence. Let $0 < t_1 < t_2$. We prove that $(W_\epsilon(t_1), W_\epsilon(t_2), B_{t_1}, B_{t_2})$ converges in law to $(\sigma W_{t_1}, \sigma W_{t_2}, B_{t_1}, B_{t_2})$. We apply decomposition (3.7) to $W_\epsilon(t_2)$.

By Point **6.a**, $(W_\epsilon(t_1), B_{t_1})$ converges in law to $(\sigma W_{t_1}, B_{t_1})$ and $(\Theta_\epsilon(t_1,t_2), B_{t_2} - B_{t_1})$ converges to $(\sigma W_{t_2} - \sigma W_{t_1}, B_{t_2} - B_{t_1})$. Since $(\Theta_\epsilon(t_1,t_2), B_{t_2} - B_{t_1})$ is independent from $(W_\epsilon(t_1), B_{t_1})$, we can conclude that $(W_\epsilon(t_1), W_\epsilon(t_2), B_{t_1}, B_{t_2})$ converges in law to $(\sigma W_{t_1}, \sigma W_{t_2}, B_{t_1}, B_{t_2})$. ∎

4 Proofs of Theorem 1.5 and Proposition 1.7

1. Convergence in distribution of a family of stochastic integrals with respect to W_ϵ.
Recall that W_ϵ is a continuous martingale, which converges in distribution to σW as $\epsilon \to 0$. Then, by Proposition 3.2 of [3], W_ϵ satisfies the condition of uniform tightness.

Consequently, from Theorem 5.1 of [3], we can deduce that for any càdlàg predictable process Λ such that (Λ, W_ϵ) converges in distribution to (Λ, W), then the process $\left(\int_0^t \Lambda_u dW_\epsilon(u)\right)_{t\geqslant 0}$ converges in distribution to $\left(\sigma \int_0^t \Lambda_u dW_u\right)_{t\geqslant 0}$ as $\epsilon \to 0$.

2. Proof of Proposition 1.7. Recall that $\Delta_\epsilon^{(2)}(H,t)$ is defined by (1.6).
First, let us consider the case $H_t = 1$ for all $t \geqslant 0$. Using Itô's formula, we obtain:

$$(B_{(s+\epsilon)\wedge t} - B_s)^2 = 2\int_s^{(s+\epsilon)\wedge t} (B_u - B_s)dB_u + (s+\epsilon)\wedge t - s.$$

Reporting in $\Delta_\epsilon^{(2)}(t)$ and applying stochastic Fubini's theorem come to

$$\Delta_\epsilon^{(2)}(1,t) = 2W_\epsilon(t) + \frac{1}{\sqrt{\epsilon}}\left(\int_0^t \frac{(s+\epsilon)\wedge t - s}{\epsilon} ds - t\right).$$

More generally, if we consider $H_s = f(B_u, u \leqslant s)$ a continuous function of the trajectory so that $t \to H_t$ is locally Hölder continuous, we can write a similar decomposition:

$$\Delta_\epsilon^{(2)}(H,t) = 2\int_0^t H_u dW_\epsilon(u) + R_\epsilon(t),$$

where
$R_\epsilon(t) = \frac{2}{\epsilon\sqrt{\epsilon}}\int_0^t \left[\int_{(u-\epsilon)^+}^u (H_s - H_u)(B_u - B_s)ds\right] dB_u + \frac{1}{\epsilon\sqrt{\epsilon}}\int_{(t-\epsilon)^+}^t H_s(t-s-\epsilon)ds.$
Since $s \to H_s$ is continuous, then a.s.

$$\lim_{\epsilon\to 0} \sup_{t\in[0,T]} \left| \frac{1}{\epsilon\sqrt{\epsilon}} \int_{(t-\epsilon)^+}^t H_s(t-s-\epsilon)ds \right| = 0.$$

Using the Hölder property of H and Doob inequality, we easily obtain

$$E\left[\sup_{t\in[0,T]} \left(\frac{2}{\epsilon\sqrt{\epsilon}}\int_0^t \left[\int_{(u-\epsilon)^+}^u (H_s - H_u)(B_u - B_s)ds\right] dB_u\right)^2\right] \leqslant C\epsilon^{2\alpha}.$$

Since $H_s = f(B_u, u \leqslant s)$ is a continuous function of the trajectory and (W_ϵ, B) converges in distribution to $(\sigma W, B)$ (cf. Theorem 1.4), applying Point **1** with $\Lambda = H$ leads to Proposition 1.7. ■

3. Proof of Point (2) **of Theorem 1.5.** Recall that $\Delta_\epsilon(H,t)$ is defined by (1.5). Since V vanishes at time 0, we prolong V to $]-\infty, +\infty[$, setting $V_s = 0$

if $s \leqslant 0$. Using $B_{(s+\epsilon)\wedge t} - B_s = \int_s^{(s+\epsilon)\wedge t} dB_u$, then Fubini's stochastic theorem yields to

$$\Delta_\epsilon(V,t) = \int_0^t \left(\frac{1}{\epsilon\sqrt{\epsilon}} \int_{u-\epsilon}^u (V_s - V_u) ds \right) dB_u.$$

Consequently,

$$E\left(\sup_{t\in[0,T]} |\Delta_\epsilon(V,t)|^2 \right) \leqslant 4 \int_0^T E\left(\frac{1}{\epsilon\sqrt{\epsilon}} \int_{u-\epsilon}^u (V_s - V_u) ds \right)^2 du.$$

Let us bound the term in parenthesis. Since V is Hölder continuous, we have

$$\left| \frac{1}{\epsilon\sqrt{\epsilon}} \int_{u-\epsilon}^u (V_s - V_u) ds \right| \leqslant \frac{1}{\epsilon\sqrt{\epsilon}} \int_{u-\epsilon}^u C\,|s-u|^\alpha\, ds \leqslant C\epsilon^{\alpha-\frac{1}{2}}.$$

Consequently,

$$E\left(\sup_{t\in[0,T]} |\Delta_\epsilon(V,t)|^2 \right) \leqslant CT\epsilon^{\alpha-\frac{1}{2}}.$$

Since $\alpha > \frac{1}{2}$, item (2) of Theorem 1.5 is proved. ■

4. Proof of Point (3) **of Theorem 1.5.**

Let $\Lambda_s = f(B_u, u \leqslant s)$ be a continuous function of the trajectory, such that $t \to \Lambda_t$ is a continuous map from $\mathbb{R}^+$ to $L^2(\Omega)$. Suppose moreover that $M_t = \int_0^t \Lambda_u dB_u$. Proceeding as in Point **3** of section 2, we have

$$\Delta_\epsilon(M,t) = \int_0^t \frac{1}{\sqrt{\epsilon}} \left[\frac{1}{\epsilon} \int_{(u-\epsilon)^+}^u M_s ds - M_u \right] dB_u.$$

Using the identity $M_u = \frac{u-(u-\epsilon)^+}{\epsilon} M_u + \frac{(\epsilon-u)^+}{\epsilon} M_u$ leads to $\Delta_\epsilon(M,t) = \Delta'_\epsilon(M,t) - R_\epsilon(t)$, where

$$\Delta'_\epsilon(M,t) = -\int_0^t \left(\frac{1}{\epsilon\sqrt{\epsilon}} \int_{(u-\epsilon)^+}^u (M_u - M_s) ds \right) dB_u,$$

and $R_\epsilon(t) = \int_0^\epsilon \frac{\epsilon-u}{\epsilon\sqrt{\epsilon}} M_u dB_u$. We claim that $R_\epsilon(t)$ is a remainder term. Indeed, Doob's inequality gives

$$E\left[\sup_{t\in[0,T]} R_\epsilon(t)^2 \right] \leqslant 4 \int_0^\epsilon \frac{(\epsilon-u)^2}{\epsilon^3} E[\int_0^u \Lambda_v dB_v]^2 du \leqslant \frac{4}{\epsilon} \int_0^\epsilon \int_0^u E[\Lambda_v^2] dv du.$$

Consequently, $R_\epsilon(t)$ does not contribute to the limit in law of $\Delta_\epsilon(M,t)$.

Using the identity $M_u - M_s = \Lambda_u(B_u - B_s) + \int_s^u (\Lambda_r - \Lambda_u) dB_r$, we decompose $\int_{(u-\epsilon)^+}^u (M_u - M_s) ds$ as

$$\begin{aligned}
&\Lambda_u \int_{(u-\epsilon)^+}^u (B_u - B_s) ds + \int_{(u-\epsilon)^+}^u \left(\int_s^u (\Lambda_r - \Lambda_u) dB_r \right) ds \\
&= \Lambda_u \int_{(u-\epsilon)^+}^u (B_u - B_s) ds + \int_{(u-\epsilon)^+}^u (r - (u-\epsilon)^+)(\Lambda_r - \Lambda_u) dB_r.
\end{aligned}$$

Consequently:

$$\Delta'_\epsilon(M,t) = -\int_0^t \Lambda_u dW_\epsilon(u) + R'_\epsilon(t),$$

where

$$R'_\epsilon(t) = -\frac{1}{\epsilon\sqrt{\epsilon}} \int_0^t \left(\int_{(u-\epsilon)^+}^u (r-(u-\epsilon)^+)(\Lambda_r - \Lambda_u) dB_r \right) dB_u.$$

It can be proved, as it is shown previously, that $R'_\epsilon(t)$ goes to 0 in $L^2(\Omega)$ as $\epsilon \to 0$. The convergence in law of $(-\int_0^t \Lambda_u dW_\epsilon(u))_{t\geqslant 0}$ to $(\sigma \int_0^t \Lambda_u dW_u)_{t\geqslant 0}$ is a direct consequence of item **1** above. ∎

5. Proof of Point (1) **of Theorem 1.5.** We have, for a fixed t and for all $\epsilon < t$:

$$\Delta_\epsilon(H_0,t) = \frac{H_0}{\sqrt{\epsilon}} \left[\frac{1}{\epsilon} \int_0^t (B_{(s+\epsilon)\wedge t} - B_s) ds - \int_0^t dB_s \right].$$

Since $B_{(s+\epsilon)\wedge t} - B_s = \int_s^{(s+\epsilon)\wedge t} dB_u$, using Fubini's Theorem allows to gather the integrals and we get

$$\Delta_\epsilon(H_0,t) = \frac{H_0}{\sqrt{\epsilon}} \left[\int_0^t \frac{u-\epsilon-(u-\epsilon)^+}{\epsilon} dB_u \right] = \frac{H_0}{\sqrt{\epsilon}} \left[\int_0^\epsilon \frac{u-\epsilon}{\epsilon} dB_u \right] = H_0 N_\epsilon,$$

where

$$N_\epsilon = \int_0^\epsilon \frac{u-\epsilon}{\epsilon\sqrt{\epsilon}} dB_u.$$

The r.v N_ϵ does not depend on t anymore and is independent from $\mathcal{F}_0$. Moreover N_ϵ has a centered Gaussian distribution, with variance

$$E(N_\epsilon^2) = \int_0^\epsilon \left(\frac{u}{\epsilon} - 1 \right)^2 \frac{du}{\epsilon} = \frac{1}{3}.$$

Finally, N_ϵ follows the Gaussian law $\mathcal{N}(0,\sigma)$. ∎

Note that $\Delta_\epsilon(H_0,0) = 0$. Consequently, the process $(\Delta_\epsilon(H_0,t))_{t\geqslant 0}$ cannot converge in distribution as $\epsilon \to 0$.

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